The Journal of Symbolic Geometry
Volume 1 (2006)

The Journal of Symbolic Geometry

Loci Of Midpoints Of Pedal Triangles

Philip Todd & Boonyapa Purt

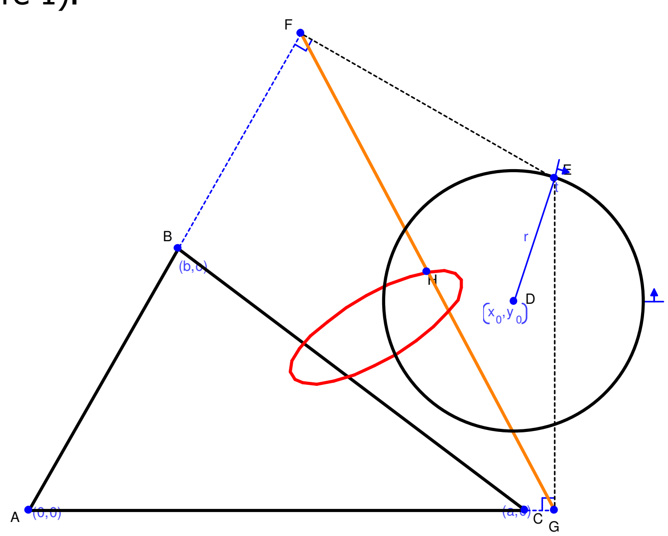
Saltire Software, PO Box 1565, Beaverton OR, USA

SouthRidge High School, Beaverton OR, USA.

Abstract: Given a reference triangle, we investigate the loci of the midpoints of the pedal triangle as the pedal point runs round a circle. These loci are ellipses whose principal axes align with the angle bisectors of the reference triangle. We show that the axes of these ellipses intersect in a common point if and only if the center of the circle lies on the line joining the incenter to the circumcenter, in which case the intersection of the axes lies on the Nagel Line.

1. Introduction

Given a triangle ABC, and a circle DE, the locus of the midpoint of a side of the pedal triangle as the pedal point runs round the circumference of DE is an ellipse (figure 1).



$$\Rightarrow -c^4 \cdot r^2 + c^4 \cdot x_0^2 + c^4 \cdot y_0^2 + Y^2 \cdot \left(16 \cdot b^4 + 20 \cdot b^2 \cdot c^2 + 4 \cdot c^4\right) + X \cdot Y \cdot \left(-16 \cdot b^3 \cdot c - 16 \cdot b \cdot c^3\right) + X^2 \cdot \left(4 \cdot b^2 \cdot c^2 + 4 \cdot c^4\right) + Y \cdot \left(4 \cdot b \cdot c^3 \cdot x_0 - 8 \cdot b^2 \cdot c^2 \cdot y_0 - 4 \cdot c^4 \cdot y_0\right) + X \cdot \left(-4 \cdot c^4 \cdot x_0 + 4 \cdot b \cdot c^3 \cdot y_0\right) = 0$$

Figure 1: Locus of midpoints of a side of a pedal triangle as the pedal point moves round a circle. The locus equation (shown) is an ellipse.

In this paper we look at the ellipses formed by the midpoints of the three sides of the pedal triangle, and ask the question, when are the axes of these ellipses concurrent?

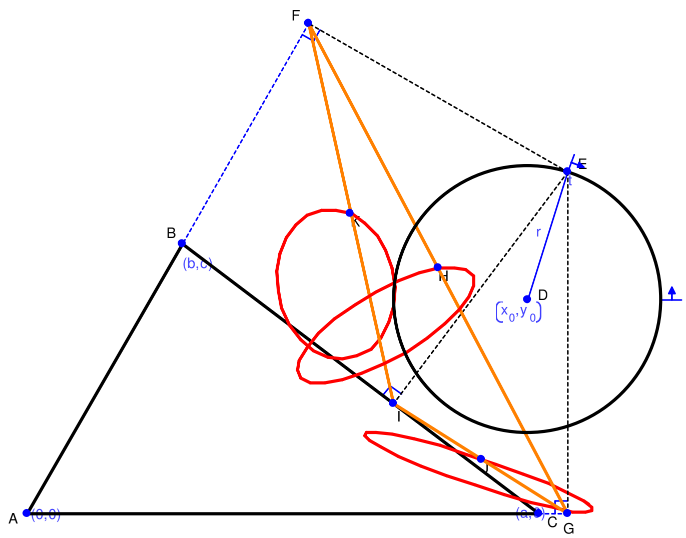


Figure 2: Loci of midpoints of a side of all three sides of the pedal triangle as the pedal point moves round a circle.

2. Ellipse Description

Definition: Given 2 non-parallel lines L_1 and L_2, we define the transformation T_{L_1,L_2} such that, for any point P, $T_{L_1,L_2}(P)$ is the midpoint of the projections of P onto L_1 and L_2 .

Lemma 1: T_{L_1,L_2} is an affine transformation with principal axis on the angle bisectors of L_1, L_2 and has principal values $\cos^2(\theta)$ $\sin^2(\theta)$, where θ is half the angle between L_1 and L_2.

Proof: If we align our axes such that the origin lies at the intersection of the lines, and the x axis lies along the perpendicular bisector, simple trigonometry gives the result (figure 3)

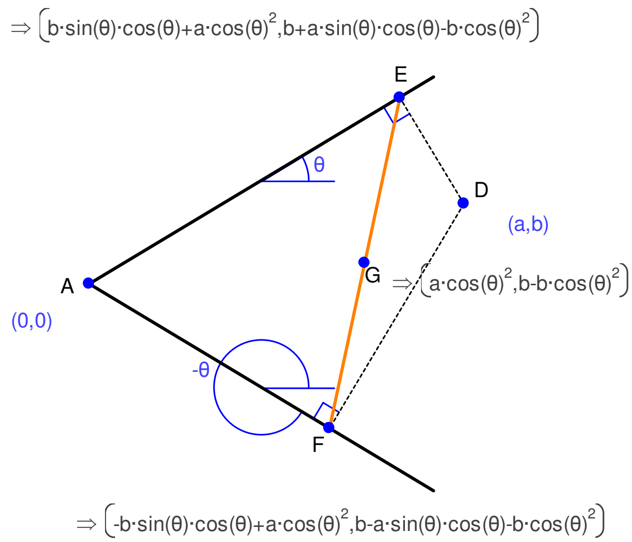


Figure 3: E and F are the projections of point(a,b) onto lines through the origin with slopes θ and –θ. G is their midpoint.

Corollary 2: If P is on the angle bisector of L_1 and L_2, then $T_{L_1,L_2}(P)$ is also on the angle bisector.

Proof: b=0 in figure 3

Theorem 3: If Ω is a circle centered at P, then $T_{L_1,L_2}(\Omega)$ is an ellipse centered at $T_{L_1,L_2}(P)$, whose principal axes align with the angle bisectors of L_1 and L_2, whose semi major and semi minor axes sum to the radius of Ω, and are in the ratio $\tan^2(\theta)$.

Proof: follows directly from Lemma 1.

3. Ellipse Triples

Notation: Given a triangle ABC, let L_1 be the line BC, L_2 be the line AC, and L_3 be the line AB. We write T_A for T_{L_2,L_3}, T_B for T_{L_1,L_3}, T_C for T_{L_1,L_2}

Given a circle Ω centered at D, we seek to characterize the conditions under which the principal axes of the ellipses $T_A(\Omega)$, $T_B(\Omega)$, $T_C(\Omega)$ meet at a common point.

Definition: Given a triangle ABC and a pedal point D, we define $L_A(D)$ to be the line through $T_A(D)$ parallel to the bisector of angle BAC.

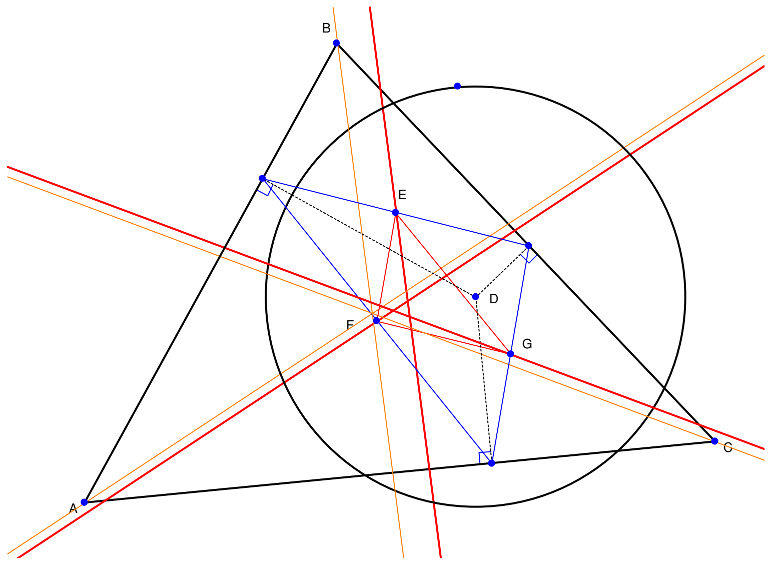


Figure 4: E,F,G are vertices of the medial triangle of the pedal triangle. We show lines parallel to the angle bisectorsof the original triangle through E,F and G. These align with principal axes of the ellipses under study. In this example, they are clearly not concurrent.

Lemma 4: The centers of the ellipses $T_A(\Omega)$, $T_B(\Omega)$, $T_C(\Omega)$ are $T_A(D)$, $T_B(D)$, $T_C(D)$ and their principal axes are $L_A(D)$, $L_B(D)$, $L_C(D)$ (and their perpendiculars)

Proof: Follows directly from the definition of T and from Theorem 3.

Lemma 5: If I is the incenter of ABC, then $L_A(I)$, $L_B(I)$, $L_C(I)$ are the angle bisectors of ABC and meet at the incenter.

Proof: I lies on the angle bisectors of all 3 angles of ABC. By lemma 2, $T_A(I)$, $T_B(I)$, $T_C(I)$ lie on the bisectors of angle A, B and C respectively..

Lemma 6: If O is the circumcenter of ABC, then $L_A(O)$, $L_B(O)$, $L_C(O)$ meet at the incenter of the medial triangle of the medial triangle of ABC. (this is the Kimberling point X(1125).

Proof: The pedal triangle of the circumcenter is the medial triangle of the reference triangle. $T_A(O)$, $T_B(O)$, $T_C(O)$ are therefore the vertices of the medial triangle of the medial triangle of ABC. As this triangle is similar to and similarly oriented to ABC, the lines $L_A(O)$, $L_B(O)$, $L_C(O)$ are the perpendicular bisectors of the triangle $T_A(O)$, $T_B(O)$, $T_C(O)$, and hence meet at its incenter.

Theorem 6: For a given triangle ABC, and point D, the lines $L_A(D)$, $L_B(D)$, $L_C(D)$ meet at a point if and only if D lies on the line IO joining the incenter and the circumcenter of ABC.

Proof: let **u**,**v**,**w** be vectors parallel to the angle bisectors of ABC. By lemma 5, there are constants $\alpha_0, \beta_0, \gamma_0$ such that

$$T_A(I) + \alpha_0 \mathbf{u} = T_B(I) + \beta_0 \mathbf{v} = T_C(I) + \gamma_0 \mathbf{w}$$

By Lemma 6, there are constants $\alpha_1, \beta_1, \gamma_1$ such that

$$T_A(O) + \alpha_1 \mathbf{u} = T_B(O) + \beta_1 \mathbf{v} = T_C(O) + \gamma_1 \mathbf{w}$$

Let P be a point on the line IO, then P can be written:

$$P = (1-k)I + kO$$

for some constant k. As T_A is affine and hence linear,

$$T_A(P) = T_A((1-k)I + kO) = (1-k)T_A(I) + kT_A(O)$$

We show that the point:

$$T_A(P) + ((1-k)\alpha_0 + k\alpha_1)\mathbf{u}$$

is common to $L_A(P)$, $L_B(P)$, $L_C(P)$.

$$T_A(P) + ((1-k)\alpha_0 + k\alpha_1)\mathbf{u} = (1-k)T_A(I) + kT_A(O) + (1-k)\alpha_0 \mathbf{u} + k\alpha_1 \mathbf{u}$$

$$= (1-k)(T_A(I) + \alpha_0 \mathbf{u}) + k(T_A(O) + \alpha_1 \mathbf{u}) = (1-k)(T_B(I) + \beta_0 \mathbf{v}) + k(T_B(O) + \beta_1 \mathbf{v})$$

$$= T_B(P) + ((1-k)\beta_0 + k\beta_1)\mathbf{v}$$

Similarly,

$$T_A(P) + ((1-k)\alpha_0 + k\alpha_1)\mathbf{u} = T_C(P) + ((1-k)\gamma_0 + k\gamma_1)\mathbf{w}$$

Hence the point $T_A(P) + ((1-k)\alpha_0 + k\alpha_1)\mathbf{u}$ lies on all 3 lines $L_A(P)$, $L_B(P)$, $L_C(P)$.

Now assume point Q not lying on OI satisfies the condition that $L_A(Q)$, $L_B(Q)$, $L_C(Q)$ meet at a common point. The vectors OI and OQ span the plane, and hence any point P can be expressed as a linear combination of O,I and Q. Hence by a similar argument to the above, $L_A(P)$, $L_B(P)$, $L_C(P)$ meet at a common point. But there do exist points P where $L_A(P)$, $L_B(P)$, $L_C(P)$ do not meet at a point (figure 4), hence no such Q exists.

Theorem 7: If P is the point I+k(O-I) on the line joining the incenter to the circumcenter, then the lines $L_A(P)$, $L_B(P)$, $L_C(P)$ meet at the point on the Nagel line $I+\frac{3}{4}k(G-I)$, where G is the centroid.

Proof: Let Y(P) map a point on the line IO onto the intersection of the lines $L_A(P)$, $L_B(P)$, $L_C(P)$. Y is linear and Y(I) = I and Y(O) = X(1125). Now X(1125) lies on the Nagel line [1], and in fact if G is the centroid, $X(1125)=I+\frac{3}{4}(G-I)$. Hence the result.

4. References

[1] K. Kimberling Encyclopedia of Triangle Centers:
http://faculty.evansville.edu/ck6/encyclopedia/